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Nonperturbative Results on the Point Particle Limit of N=2 Heterotic String Compactifications

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Using heterotic/type II string duality, we obtain exact nonperturbative results for the point particle limit ($\alpha' \to 0$) of some particular four dimensional, $N=2$ supersymmetric compactifications of heterotic strings. This allows us to recover recent exact nonperturbative results on $N=2$ gauge theory directly from tree-level type II string theory, which provides a highly non-trivial, quantitative check on the proposed string duality. We also investigate to what extent the relevant singular limits of Calabi-Yau manifolds are related to the Riemann surfaces that underlie rigid $N=2$ gauge theory.

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Recently there have been exciting developments in understanding non-perturbative aspects of string theory through conjectured string dualities [1]. In particular, the geometry of moduli spaces of $N = 1, 2$ and 4 supersymmetric string vacua is getting better understood. Since for $N = 4$ the geometry of the moduli space is uncorrected even non-perturbatively, the $N = 1, 2$ cases are much more interesting, as far as shedding light on non-perturbative dynamics of string theory is concerned. This is also mirrored in the interesting dynamics of the $N = 1, 2$ field theories [2,3,4,5,6]. In this paper we show how some of the exact results on the quantum moduli space of certain $N = 2$ string vacua [7] can reproduce in the point particle limit (where $M_{planck} \to \infty$) the exact field theory results of [2].[1] This provides a truly nonperturbative, quantitative check on the proposed heterotic/type II string duality.[2]

We will concentrate on the two main models studied in [7], for which there have already been many non-trivial checks in perturbation theory [13,14,15]. We will first study in some detail the rank three model of [7], which we will call model A, and then discuss how our results generalize to the second main model of [7] (the rank four model, which we will call model B). More details, especially concerning the string and gravitational contributions to the exact nonperturbative effective action, will appear in a subsequent paper [16].

1. Description of Model A

Model A has two equivalent descriptions: We can view it as the $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$, where we choose the T and U moduli of the two-torus to be equal so that there is an extra $SU(2)$ gauge symmetry. We also choose the second Chern class of the $E_8 \times E_8 \times SU(2)$ gauge bundle to be $(10, 10, 4)$, giving a total of 24 that equals the second Chern class of $K3$; this is required for world-sheet anomaly cancellations. This model has 129 hypermultiplets whose scalars characterize the geometry of $K3$ with the corresponding bundles on it, and 2 vector multiplets whose scalars give the modulus T of the two-torus and the dilaton/axion field S. The dual description of this model is given

[1] The question of going to the point particle limit has been addressed before in [8,9,10,11,12].

[2] It also substantiates the conjectures that the local analogs of the Seiberg-Witten Riemann surfaces are given by Calabi-Yau manifolds [8,9] and that space-time Yang-Mills instanton effects can be described in terms of world-sheet instantons of type II string theory [4].

by a type IIB (or type IIA) string compactification on a Calabi-Yau manifold M (or its mirror), with defining polynomial

$$p = {z_1}^{12} + {z_2}^{12} + {z_3}^6 + {z_4}^6 + {z_5}^2 - 12\psi\, z_1 z_2 z_3 z_4 z_5 - 2\phi\, {z_1}^6 {z_2}^6 \ , \tag{1.1}$$

where z_i are coordinates of $WP^{12}_{1,1,2,2,6}$ and where we mod out by all phase symmetries that preserve the holomorphic three-form. This Calabi-Yau manifold has $h_{11} = 128$ and $h_{21} = 2$, giving rise to 2 vector multiplets (whose scalar expectation values correspond to ϕ and ψ above) and 129 hypermultiplets (including the type II string dilaton). If we wish to study the moduli space of vector multiplets, tree-level type II string theory is exact, whereas if we wish to study the moduli space of hypermultiplets, the tree level of the heterotic side is exact. In this paper we consider the moduli space of vector multiplets, and so we study the classical moduli space of the type II side spanned by ϕ and ψ in the above defining equation.

The classical moduli space of this model has been studied in great detail in [17,18]. It is convenient to introduce the variables

$$x = -\frac{1}{864}\frac{\phi}{\psi^6} \ , \qquad y = \frac{1}{\phi^2} \ . \tag{1.2}$$

According to the identification of [7], the T and S fields of the heterotic side should be identified (in the large S/weak coupling regime) with the special coordinates corresponding to x and y, respectively. In particular, for large S one has:

$$x = \frac{1728}{j(T)} + \dots \ , \qquad y = \exp(-S) + \dots \tag{1.3}$$

This identification was in part motivated by the fact that at $T = i$ the perturbative heterotic model develops an $SU(2)$ gauge symmetry. The existence of the $SU(2)$ gauge symmetry of heterotic strings is reflected by the existence of the conifold locus of M, which is given by

$$\Delta \ = \ (1-x)^2 - x^2 y \ = \ 0 \ . \tag{1.4}$$

For weak coupling, $y \to 0$, there is a double singularity at $x = 1$ (corresponding to $T = i$). Moreover, for finite coupling corresponding to finite y, there are two singular loci for x, in line with the field theory results of [2] where one has two singular points in the moduli space associated with massless monopoles/dyons.

2. What to Expect when Gravity is Turned Off?

It would be a very non-trivial test of all these ideas if we could show that in the limit of turning off gravitational/stringy effects, we would reproduce the results of [2], where the quantum moduli space of pure $N=2$ Yang-Mills theory with $SU(2)$ gauge group has been studied. This corresponds to considering the point particle limit of strings obtained by taking $\alpha' \to 0$. To this end note that the variable $u = tr\phi^2$ that vanishes at the $SU(2)$ point should, to leading order, be identified with $x-1$. To make this dimensionally correct, we must have

$$x = 1 + \alpha' u + O(\alpha')^2 \ . \tag{2.1}$$

Note that as $\alpha' \to 0$, the full u-plane is mapped to an infinitesimal neighborhood of $T=i$. This in particular means that the effect of the modular geometry of T is being turned off in this limit, as one expects. Furthermore, in order for the scale Λ of the $SU(2)$ theory to satisfy $\Lambda \ll M_{planck} \sim 1/\sqrt{\alpha'}$, we should tune the string coupling constant (which is defined naturally at the string scale) to be infinitesimally small. Taking into account the running of the $SU(2)$ gauge coupling constant, we should take, to leading order in $\alpha' \to 0$:

$$y = \exp(-S) = {\alpha'}^2 \Lambda^4 \exp(-\hat{S}) \ \equiv: \ \epsilon^2. \tag{2.2}$$

Thus, by dimensional transmutation the coupling constant of $SU(2)$, $e^{-\hat{S}}$, can be traded with the scale Λ, at which the $SU(2)$ gauge theory becomes strongly coupled. Note that the conifold locus (1.4) in the limit $\alpha' \to 0$ goes to

$$u^2 = \Lambda^4 \exp(-\hat{S}) \ , \tag{2.3}$$

which is the expected behaviour.

Let us recall that $N=2$ supergravity moduli are characterized by a prepotential F, which in our case is a function of T, S. Using the axionic shift symmetry, it is easy to see that it has an expansion of the form [8]

$$F = ST^2 + \sum_{n=0}^{\infty} f_n(T) \exp(-nS) \ , \tag{2.4}$$

where f_0 corresponds to one-loop string corrections and where $f_n(T)\exp(-nS)$ is the contribution from the n-th stringy instanton sector. This expansion is most convenient when

we are dealing with large T. Since we are interested in T near i, it is more convenient to shift to $\tilde{T} \sim (T-i)/(T+i)$ [19], and consider another expansion of F given by

$$F = S\tilde{T}^2 + \sum_{n=0}^{\infty} g_n(\tilde{T})\exp(-nS) + Q(S,\tilde{T}) \tag{2.5}$$

where $Q(S,\tilde{T})$ is some polynomial of first degree in S, and where we have chosen $g_0 \sim \tilde{T}^2\log\tilde{T} + O(\tilde{T}^3)$. We now consider turning off gravity by taking the limit $\alpha' \to 0$. Note that since both $\tilde{T}$ and the variable a defined in [2] are good special coordinates and are proportional to leading order, they have to be identified via

$$\tilde{T} = \sqrt{\alpha'}\, a \ . \tag{2.6}$$

This is consistent[3] with (2.1), and also correctly translates the modular transformation $T \to -1/T$ to the Weyl transformation $a \to -a$ [9,20,21,12]. Now using (2.2) and (2.6) we reexpand (2.5) and get

$$F = \alpha'\hat{S}a^2 + \sum_{n=0}^{\infty} g_n(\sqrt{\alpha'}a){\alpha'}^{2n}\Lambda^{4n}\exp(-n\hat{S}) + Q(\hat{S},\sqrt{\alpha'}a) \ . \tag{2.7}$$

In order to recover the results of Seiberg and Witten as $\alpha' \to 0$, we must find for large a:

$$g_n(\sqrt{\alpha'}a) = c_n(\sqrt{\alpha'}a)^{2-4n} + O((\sqrt{\alpha'}a)^{2-4n+1}) \ , \tag{2.8}$$

where c_n are the instanton coefficients of [2] in the weak coupling regime. In other words, let $F_{SW}(a,\Lambda^4)$ be the prepotential obtained in [2]. Then, if g_n behaves as above, we would have

$$F(a,\hat{S}) = \alpha' F_{SW}(a,\Lambda^4\exp(-\hat{S})) + \tilde{Q}(\hat{S},\sqrt{\alpha'}a) + O(\alpha')^{3/2} \ , \tag{2.9}$$

where $\tilde{Q}$ is a quadratic polynomial of first degree in S and second degree in $\sqrt{\alpha'}\,a$. In order to compare the above result with the periods of M that we will determine in this limit below, it is useful to recover, using special geometry, the periods from the prepotential F

[3] To be precise, in order to recover the conventional definition of a in relation to u, note that there is a proportionality constant in this equation related to the second order expansion coefficient of the j-function near $T = i$. We can avoid this by rescaling u in the definition of (2.1) and redefining Λ in such a way that $\tilde{u} = u/(\Lambda^2 e^{-\hat{S}/2})$ is invariant. This will have no effect on the equations below.

given in (2.9). Since we are working in a non-homogeneous basis, the A-type periods can be taken to be (proportional to)

$$(1, S = \hat{S} - \log(\Lambda^4 \alpha'^2), \sqrt{\alpha'} a) \ , \tag{2.10}$$

while the corresponding B-type periods are given by

$$(2F - S\partial_S F - a\partial_a F, \partial_S F, \frac{1}{\sqrt{\alpha'}}\partial_a F) \ . \tag{2.11}$$

This simplifies due to the remarkable fact (whose full physical significance remains to be uncovered) proven in [22] that

$$\pi i (F_{SW} - \frac{1}{2} a \partial_a F_{SW}) = (-2\pi i) \partial_S F_{SW} = u \ . \tag{2.12}$$

This is crucial for us in obtaining the rigid theory in the limit of turning off gravity. Using (2.12) we find that the B periods must be certain linear combinations of the A periods with $\sqrt{\alpha}' a_D, \alpha' u, \alpha' u \hat{S}$. Thus, up to linear combinations, we should get the following 6 periods

$$(1, S, \sqrt{\alpha'} a, \sqrt{\alpha'} a_D, \alpha' u, \alpha' u S) \ . \tag{2.13}$$

We will verify below that the periods of the Calabi-Yau manifold M in the limit $\alpha' \to 0$ are indeed given by linear combinations of the above six periods.

3. Geometrical Characterization of the Appearance of (a, a_D)

Before going on in the next section to solve the Picard-Fuchs equations and obtain the six periods that we expect to emerge in the point particle limit, we would like to give a geometrical idea of how the two most interesting periods, namely the rigid periods $a(u), a_D(u)$ of [2], appear. Given the fact that in [2] a, a_D were periods of a meromorphic one-form on a torus, it is important to see, by a means more transparent than direct computation, how this geometrical structure is encoded in the Calabi-Yau manifold M in the vicinity of $y = 0$, $x = 1$.

As we approach the conifold locus in moduli space, some three-cycle is shrinking to zero size. We expect that the computation of (a, a_D) is only affected by integrals localized in the neighborhood of the collapsing cycle. Therefore, we should try to understand the appearance of the rigid periods (a, a_D) by approaching $y = 0$, $x = 1$ in moduli space and

by simultaneously rescaling variables to "blow up" a neighborhood of the singular locus on the manifold M.

From the above we know that as we approach the point of interest, the moduli of M scale as[4]

$$\phi = -\frac{1}{\epsilon}\ , \qquad \psi = -\frac{1}{\sqrt{3}}\left(2^5\epsilon\,(1+\epsilon\tilde{u})\right)^{-1/6}\ . \tag{3.1}$$

If we want to keep the conifold singularity at a finite point in our rescaled variables, fixing z_1 it turns out that the unique choice of rescaling is $\tilde{z}_{3,4} = \epsilon^{1/6} z_{3,4}$, $\tilde{z}_5 = \epsilon^{1/2} z_5$. Then, if we in addition define $\zeta = z_1 z_2$, and rescale z_i by irrelevant numerical factors, we find that the defining equation $p = 0$ of M can be rewritten as

$$p_0 = \frac{1}{6}\tilde{z}_3{}^6 + \frac{1}{6}\tilde{z}_4{}^6 + \frac{1}{6}\zeta^6 + \frac{1}{2}\tilde{z}_5{}^2 + \zeta\,\tilde{z}_3\tilde{z}_4\tilde{z}_5 = \epsilon \tag{3.2}$$

$$p_1 = \frac{1}{12}\,z_1{}^{12} + \frac{1}{12}\,z_2{}^{12} + \frac{\tilde{u}}{6} z_1 z_2 \tilde{z}_3\tilde{z}_4\tilde{z}_5 = -1\ . \tag{3.3}$$

Here we have simply used that as $\epsilon \to 0$, the defining polynomial can be written as $p = \frac{1}{\epsilon}p_0 + p_1$.

As ϵ goes to zero, the leading singularity is described by $p_0 = \epsilon$. However, note that this is itself a singular space! It is quite clear that the periods that we are interested in are governed by the subleading piece (3.3), which smooths out the singularity in (3.2) for finite ϵ. More concretely, we are suggesting that the periods related to three-cycles that are not collapsing as we approach $y = 0, x = 1$ should be controlled by the leading term in the ϵ-expansion. On the other hand, the $\tilde{u}$-dependent periods, a, a_D, are governed by the sub-leading term, p_1, which is the first $\tilde{u}$-dependent term in the ϵ-expansion. Therefore, in order to study the periods a, a_D, it should be enough to focus on the variation of p_1 with $\tilde{u}$. Furthermore, since we are interested in the leading behavior in the ϵ expansion, we can solve for $\tilde{z}_{3,4,5}$ in (3.2) at the singularity. This will be more fully justified below.

Thus solving for the singular locus in $p_0 = 0$, we find $\tilde{z}_{3,4} = z_1 z_2$, $\tilde{z}_5 = -(z_1 z_2)^3$, and substituting this into p_1, we see that the manifold whose Hodge variation must give (a, a_D) is the curve

$$\frac{1}{12}\,z_1{}^{12} + \frac{1}{12}\,z_2{}^{12} - \frac{1}{6}\tilde{u}\,z_1^6 z_2^6 \;=\; 1\ . \tag{3.4}$$

We notice that this curve is very similar to the following $\Gamma(2)$ torus in $WP^3_{1,1,2}$ (in the patch where $w_3 = 1$),

$$\frac{1}{4}w_1{}^4 + \frac{1}{4}w_2{}^4 - \frac{1}{2}\tilde{u}\,w_1^2 w_2^2 \;=\; w_3{}^2\ , \tag{3.5}$$

[4] In the following, we use the dimensionless variable $\tilde{u} \equiv u/(\Lambda^2 e^{-\hat{S}/2})$.

which underlies the rigid $SU(2)$ $N=2$ gauge theory [2]. Note, in particular, that the two curves share the same discriminant locus, $\tilde{u}=\pm 1$. One may in fact view (3.4) as a triple cover of the Seiberg-Witten torus.[5]

Obtaining the geometrical torus, or more precisely a geometrical object with equivalent Hodge variation, is not quite enough to yield the periods a and a_D. Another ingredient in [2] was the choice of a particular *meromorphic* one-form λ, whose periods are a and a_D. The meromorphic one-form has no residue and satisfies $\partial_u \lambda = \omega$, where ω is the holomorphic one-form on the torus. How does this emerge for us? In order to address that and make our discussion of the limit $\epsilon \to 0$ above somewhat more rigorous, we use the definition of periods given in [23,24]. That is, we write the periods of M as

$$\Pi_i = \int_{\gamma_i} dz_1\, dz_2\, d\tilde{z}_3\, d\tilde{z}_4\, d\tilde{z}_5\, \exp(iW) \ , \tag{3.6}$$

where W is given by the defining polynomial in weighted projective space, and γ_i are a basis for an appropriate class of cycles [23]. In our case, $W = \frac{p_0}{\epsilon} + p_1$. We can now reformulate what we were doing before: In the limit as $\epsilon \to 0$ the leading contribution to the integral comes from going to the saddle points of p_0 (this is of course nothing but the stationary phase method). Thus we have to find the 'minima of the action', which means solving

$$\partial_i p_0 = 0 \ .$$

The minima of the action are not isolated, because $p_0 = 0$ gives a singular manifold. Shifting

$$\hat{z}_{3,4} = \tilde{z}_{3,4} - \zeta \ , \qquad \hat{z}_5 = \tilde{z}_5 + \zeta^3 \ ,$$

we find that

$$\frac{1}{\epsilon} p_0 = \frac{1}{\epsilon}\left[\frac{15}{6}\zeta^4(\hat{z}_3^2 + \hat{z}_4^2) - \zeta^4 \hat{z}_3 \hat{z}_4 + \frac{1}{2}\hat{z}_5^2 + \zeta^2(\hat{z}_3\hat{z}_5 + \hat{z}_4\hat{z}_5) + O(\hat{z}_i^3)\right]$$

It is easy to see that for fixed $\zeta \neq 0$ the 'mass matrix' for $\hat{z}_{3,4,5}$ has nonvanishing determinant. Thus the 'fields' $\hat{z}_{3,4}, \hat{z}_5$ for generic ζ are infinitely massive[6] as $\epsilon \to 0$. We can therefore integrate them out, and this results, in leading order, in substituting $\hat{z}_{3,4} = \hat{z}_5 = 0$ in

[5] A related point was made in [11] where it is observed that a triple cover of (3.5) (for $\tilde{u}=0$) appears as the locus $z_3 = z_4 = 0$ in M.

[6] This is true for all ζ except $\zeta = 0$ – it may be that the other four periods are related to this contribution.

p_1, which is precisely what we did above. From the gaussian integration we get in addition a factor of ζ^{-4}, so that

$$\Pi = \int (z_1 z_2)^{-4}\, dz_1\, dz_2\, \exp(i[\frac{1}{12}z_1^{12} + \frac{1}{12}z_2^{12} - \frac{1}{6}\tilde{u} z_1^6 z_2^6]) \ .$$

In order to make contact with geometry, we are at liberty to add an extra $\tilde{u}$-independent integral, since multiplying Π's by overall constants is irrelevant. We thus choose

$$\Pi = \int (z_1 z_2)^{-4}\ v_3^2\ dz_1\, dz_2\, dv_3\ \exp(i[\frac{1}{12}z_1^{12} + \frac{1}{12}z_2^{12} - \frac{1}{6}\tilde{u} z_1^6 z_2^6 + v_3^2])$$

(the choice of the v_3^2 term in front of the measure will be explained momentarily). In the computations of the periods we are free to make any birational transformation, so we choose $v_1 = z_1^3$, $v_2 = z_2^3$, and obtain (after irrelevant rescaling of variables)

$$\Pi = \int (v_1 v_2)^{-2}\, v_3^2\ dv_1\, dv_2\, dv_3\ \exp(i[v_1^4 + v_2^4 - 2\tilde{u} v_1^2 v_2^2 + v_3^2])$$

Note that the choice of v_3^2 above makes the factor in front scale invariant, and it is thus a function on the resulting elliptic curve. We then find that the above period is the same as that of the following meromorphic one-form (by going to the $v_2 = 1$ patch and rewriting $v_1 \to x$, $v_3 \to y$):

$$\begin{aligned} \lambda &= \frac{y^2}{x^2}\frac{dx}{y} \\ y^2 &\equiv x^4 - 2\tilde{u}x^2 + 1\ . \end{aligned} \tag{3.7}$$

Note that this meromorphic one-form has only second order poles, so its periods make sense, and also that

$$\frac{\partial \lambda}{\partial \tilde{u}} = \frac{dx}{y}$$

which is a necessary requirement [2]. In fact, these two ingredients fix λ up to an exact form. After the transformation [25] which brings the quartic torus (3.7) and the cubic torus of [2] into Weierstrass form one can show that λ differs from the meromorphic one form of [2] only by an exact piece. Thus a, a_D computed from the periods of λ agree with those of [2].

4. Specialization of Picard–Fuchs Operators

In the previous section we have seen how two of the periods of M reproduce the expected point particle limit periods (a, a_D). In this section we will show, by solving the Picard–Fuchs equations, that all six periods have indeed the expected form (2.13).

Solutions of the relevant Picard-Fuchs equations have been computed in [18], and provide the instanton-corrected period vector in the large complex structure limit $x = 0,\ y = 0$ $(T \to i\infty, S \to \infty)$. However, for comparison with the rigid $SU(2)$ Yang-Mills theory, we need to expand the periods around $x = 1,\ y = 0$ $(T = i, S \to \infty)$, a point which is outside the radius of convergence of the solutions given in [18]. Therefore, we will make a variable transformation to solve the Picard–Fuchs system directly in variables centered at $x = 1,\ y = 0$. This is the point of tangency between the conifold (monopole) locus, $\Delta = 0$, and the weak-coupling line, $y = 0$ (see [17] for details of the moduli space).

In order to obtain appropriate solutions in form of ascending power series, partly multiplied by logarithms, we have to be careful in the choice of variables. A proper way to do that is to blow up twice the point of tangency by inserting P^1s. This leads to divisors with only normal crossings, and the associated variables will automatically lead to solutions of the desired form. More precisely, as shown in Fig.1, the blow-up introduces two exceptional divisors, E_2 and E_3. The latter can be associated with the $SU(2)$ Yang-Mills quantum moduli space, which is given by the u plane. It intersects with the other divisors at the points $\tilde{u} = \infty$, $\tilde{u} = \pm 1$ and $\tilde{u} = 0$, corresponding to the semi-classical limit, the massless monopole points and the Z_2 orbifold point, respectively.

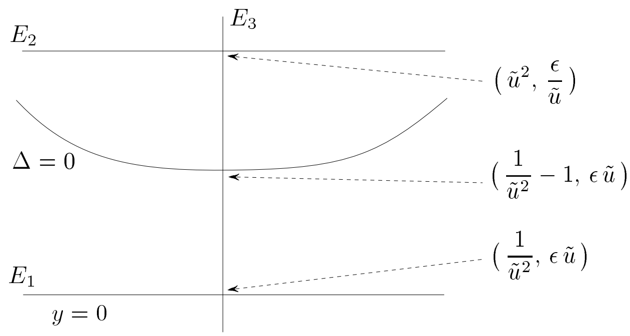

Fig.1 *The double blow-up of the intersection of the conifold locus $\Delta = 0$ with $y = 0$ leads to three divisor crossings and thus to three canonical pairs of expansion variables (x_1, x_2). They describe the physical regimes of the Seiberg-Witten theory at $\tilde{u} = 0$, $\tilde{u} = \pm 1$ and $\tilde{u} = \infty$, respectively.*

To recover the rigid periods $a(\tilde{u}), a_D(\tilde{u})$, we consider for the time being the specialization of the Picard–Fuchs system only in the semi-classical regime, $\tilde{u} \to \infty$, which corresponds to the intersection of E_1 with E_3; the other two regimes can be treated in a completely analogous way. The appropriate variables are $x_1 = x^2 y/(x-1)^2 = 1/\tilde{u}^2$ and $x_2 = (x-1) = \alpha' u \equiv \epsilon\, \tilde{u}$; in particular, x_2 is the variable which we will send to zero when we turn off gravity. After transforming the Picard–Fuchs system [17,18] to these variables, we find four solutions with index $(0,0)$ and two solutions with index $(0,1/2)$, where the index is defined by the lowest powers of the variables (modulo integers) that appear in the solution. From the monodromy around $\tilde{u} = \infty$ it is clear that the solutions related to the rigid $SU(2)$ periods must be those with index $(0,1/2)$. Specifically, we find for the first Picard–Fuchs operator in the limit $x_2 \to 0$ the following leading pieces:

$$
\begin{aligned}
\mathcal{L}_1 \ \sim\ & 288\, {x_1}^2 \partial_2\, {\partial_1}^2 x_2 + 288\, x_1\, x_2\, \partial_2\, \partial_1 - 144\, {x_1}^2 {\partial_1}^2 + 72\, {x_2}^3 {\partial_2}^3 - 216\, x_1\, \partial_1 \\
& - 288\, x_1 {x_2}^2\, \partial_1\, {\partial_2}^2 + 108\, {x_2}^2 {\partial_2}^2\ .
\end{aligned}
$$

Note that this vanishes identically when applied to a function of the form $\sqrt{x_2} f(x_1)$. On the other hand, after rescaling the solutions by ${x_1}^{1/4} {x_2}^{1/2}$ (which is motivated by the form of the solutions given below), the second PF operator becomes:

$$
\mathcal{L}_2 \ =\ -1 + 24\ (x_1 - 1)\ x_1\, \partial_1 + 16\ (x_1 - 1)\ {x_1}^2\, {\partial_1}^2 - 16\, {x_1}^2\, x_2\, \partial_1\, \partial_2 + 4\, x_1\, {x_2}^2\, {\partial_2}^2\ .
$$

For $x_2 \to 0$, this becomes precisely the PF operator $\tilde{\mathcal{L}}$ of the rigid $SU(2)$ theory [26] that has a, a_D as solutions ! [7] Note, in addition, that for $x_2 \to 0$, $\mathcal{L}_2 \cdot \lambda = 0$ (modulo an exact form), confirming our choice of meromorphic one-form in (3.7).

Having thus explicitly shown that the Seiberg-Witten periods $a(u), a_D(u)$ appear as solutions of the Calabi-Yau Picard-Fuchs system in the limit $x_2 \to 0$, it remains to verify that the structure of the full Calabi–Yau period vector is consistent with our physics expectations. Indeed, the leading terms of the six solutions in the limit $x_2 \to 0$ are given

[7] From the form of solutions given below one can see that there are no logarithms of x_2 in the relevant solutions, so that the x_2 dependence in $\mathcal{L}_2$ cannot cancel out when acting on them. Note also that $\tilde{\mathcal{L}}$ is obtained from the differential operator $\mathcal{L}$ that acts on the ordinary torus periods $(\omega, \omega_D) \equiv (\partial_u a, \partial_u a_D)$ by $\partial_u \tilde{\mathcal{L}} = \mathcal{L} \partial_u$.

by:

$$
\begin{aligned}
1+\mathcal{O}(x_2^2) &= 1+\mathcal{O}(\alpha'^2u^2)\\
x_2+O(x_2^2) &= \alpha' u+\mathcal{O}(\alpha'^2u^2)\\
\sqrt{x_2}(1+\mathcal{O}(x_2))(1-\frac{1}{16}x_1-\frac{15}{1024}x_1^2+...) &= \sqrt{\alpha'}a(\tilde{u}^2)(1+\mathcal{O}(\alpha' u))\\
(1+\mathcal{O}(x_2^2))\ln(x_1x_2^2) &= -S(1+\mathcal{O}(\alpha'^2u^2))\\
x_2(1+\mathcal{O}(x_2))\ln(x_1x_2^2) &= -\alpha' u\ S(1+\mathcal{O}(\alpha' u))\\
\sqrt{x_2}(1+\mathcal{O}(x_2))(1-\frac{1}{16}x_1-\frac{15}{1024}x_1^2+...)\ln(x_1) &= \sqrt{\alpha'}a_D(\tilde{u}^2)(1+\mathcal{O}(\alpha' u))\ ,
\end{aligned}
\tag{4.1}
$$

in perfect accordance with (2.13) ! (We intend to present the precise linear combinations that correspond to the geometric periods elsewhere [16].) Note that the appearance of odd powers of u signals the breaking of the discrete Z_8 R-symmetry of the rigid Yang-Mills theory to Z_4. This is due to the string winding modes, which break this symmetry already in string perturbation theory.

5. Calabi–Yau Monodromies and the Heterotic Duality Group

In the previous section we have solved the Picard–Fuchs equations near $x=1, y=0$. We would now like to find the monodromies of the Calabi-Yau manifold, which represent non-perturbative quantum symmetries from the viewpoint of the heterotic string. In particular, we will show that certain monodromies reproduce the monodromies of the quantum $SU(2)$ Yang-Mills theory and thus underlie the Riemann-Hilbert problem whose solution is given by the Seiberg-Witten periods, $a(u), a_D(u)$.

The calculation of the monodromy generators is completely analogous to that of the octic discussed in [17], and we refer to this paper for details and notation. In summary, the monodromy group is generated by three elements, denoted by $\mathbf{A}$,$\mathbf{B}$, and $\mathbf{T}$, which are obtained by loops in the moduli space around the $\mathbf{Z}_{12}$ identification singularity $\psi=0$, the strong coupling singularity $y=1$ and the conifold singularity $\Delta=0$, respectively.

The period vector in an integral basis is determined by the holomorphic prepotential F as $\Pi=(2F-\sum_i t_i\partial_{t_i}F, \partial_{t_1}F, \partial_{t_2}F, 1, t_1, t_2)$. By fixing the integral basis as in [27,17,28], we find in the large complex structure limit the following prepotential:

$$
F=-\frac{2}{3}t_1^3-t_1^2t_2+b_1t_1+b_2t_2+c\ +\ \text{inst. corr.}\ , \tag{5.1}
$$

where t_1 and t_2 are inhomogeneous special coordinates, defined as quotients of the periods $t_i := \frac{\omega_i(x,y)}{\omega_0(x,y)}$. Here, $\omega_0(x,y)$ is the unique power series solution in the domain around $(x,y)=0$, while $\omega_1(x,y)$, $\omega_2(x,y)$ are the unique solutions of the form $\omega_0 \log(x)+x+\ldots$, $\omega_0\log(y)+y+\ldots$. More precisely, via mirror symmetry t_1 and t_2 are the complexified parameters of the Kähler classes J_1 and J_2 that generate the Kähler cone and that correspond to the divisors in the linear system of degree two monomials and degree one monomials. As a consequence, the cubic part[8] of F is fixed by the classical intersection numbers and is given by $-\frac{1}{6}(\int_M J_i\wedge J_j\wedge J_l)t_it_jt_l$.

The Kähler structure parameters can be related to the heterotic moduli by $t_1 = T,\ t_2 = -\frac{1}{2\pi i}S$. We now identify the generators that correspond to the semi-classical heterotic duality symmetries, namely to T-duality and to the dilaton shift. The shift generators $\mathbf{T}_i:\ t_i\to t_i+1$ are immediately determined by the large complex structure limit to be $\mathbf{T}_1=(\mathbf{ATAT})^{-1},\ \mathbf{T}_2=(\mathbf{ATB})^{-1}$, whereas a generator respecting the weak coupling limit and acting as $\mathbf{S}_1: t_1\to -1/t_1$ on the semi-classical period vector is given by $\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{T}^{-1}\mathbf{A}$. A generator that acts on the special coordinates in a particularly interesting[9] way is given by $\mathbf{U}=\mathbf{ATB}^{-1}\mathbf{T}^{-1}\mathbf{A}^{-1}:\ t_1\to t_1+t_2,\ t_2\to -t_2$.

An explicit matrix representation of the generators of the duality group in the large complex structure basis is given by:

$$\mathbf{A}=\begin{pmatrix} -1 & 0 & 1 & -2 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 0\\ -1 & 1 & -1 & -1 & 2 & 1\\ 1 & 0 & 0 & 1 & 0 & 0\\ -1 & 0 & 0 & -1 & 1 & 1\\ 1 & 0 & 0 & 1 & 0 & -1\end{pmatrix},\quad \mathbf{T}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}, \tag{5.2}$$

$$\mathbf{T}_2=\begin{pmatrix} 1 & 0 & -1 & 2 & 0 & 0\\ 0 & 1 & 0 & 0 & -2 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 1\end{pmatrix}.$$

[8] The constants b_i, c are related to topological invariants [28] by $b_i=\frac{1}{24}\int_M c_2\wedge J_i$ and [27] $c=i\frac{\zeta(3)}{(2\pi)^3}\int_M c_3$, i.e. in this case $b_1=\frac{13}{6}$, $b_2=1$ and $c=-\frac{i\zeta(3)}{2\pi^3}63$.

[9] As was pointed out in [13], this transformation leads to a symmetry that is quite mysterious from the point of view of heterotic strings, if we choose the following, alternative identification: $t_2=\tilde{S}-T$. For this identification, $\mathbf{U}$ simply acts as an exchange of $\tilde{S}\leftrightarrow T$!

To make contact with the results of [2], we make a further symplectic change of basis to the string frame introduced in [9], which is characterized by a semi-classical period vector of the form

$$\begin{aligned}\Pi_{string} \;\cong\; & (-\frac{1}{2}T^2+\frac{1}{2},\;\; -T,\;\; -\frac{1}{2}T^2-\frac{1}{2},\;\; -\frac{2}{3}T^3-T^2S-\frac{13}{6}T+S-2c, \\ & -2T^2-2TS+\frac{13}{6},\;\; \frac{2}{3}T^3+T^2S+\frac{13}{6}T+S+2c)\;.\end{aligned} \tag{5.3}$$

In this basis, the monodromies $\mathbf{T}_1, \mathbf{T}_2$ and $\mathbf{S}_1$ read

$$\mathbf{T}_1 = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & \frac{3}{2} & 0 & 0 & 0 \\ -5 & 2 & 5 & \frac{1}{2} & 1 & -\frac{1}{2} \\ -2 & 4 & 2 & -1 & 1 & -1 \\ 5 & -2 & -5 & \frac{1}{2} & -1 & \frac{3}{2} \end{pmatrix},\; \mathbf{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \end{pmatrix}, \tag{5.4}$$

$$\mathbf{S}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & -1 \end{pmatrix}$$

Note that $\mathbf{S}_1 = \mathbf{M}_\infty$ contains the $SU(2)$ monodromy [2] at $u=\infty$ in an unexpectedly simple way: the non-trivial 2×2 matrix acting on the 3rd and 6th entry of the period vector is precisely M_∞ of the rigid gauge theory, in a basis with periods $(\partial_u a, 2\,\partial_u a_D)$. In fact, we can do better and determine also the strong coupling (monopole) monodromies from the fact that the monodromy around the conifold locus is $\mathbf{T}$. In this way, we find $\mathbf{M}_1 = \mathbf{T}^{-1}$, $\mathbf{M}_{-1} = (\mathbf{A}^{-1}\mathbf{T}\mathbf{A})^{-1}$ with $\mathbf{M}_1\mathbf{M}_{-1} = \mathbf{M}_\infty$, where $\mathbf{M}_1$ and $\mathbf{M}_{-1}$ are 6×6 matrices with the $SU(2)$ monopole monodromies M_1 and M_{-1} in the $(3,6)$ entries as the only non-trivial elements.

The identification of the semi-classical heterotic string monodromies, as well as of non-perturbative monodromies of the field theory limit as part of the Calabi–Yau monodromies, provide a non-trivial check on the type II-heterotic string duality. More importantly, we get a prediction for the non-perturbative duality group of the heterotic theory: it is generated

by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{T}$, subject to certain relations that are implied by the large complex structure limit [17,28], and by the Van–Kampen relations. Specifically, one can show that

$$[\mathbf{B},(\mathbf{AT})^2]=[\mathbf{B},\mathbf{A}^2\mathbf{T}]=[\mathbf{T},\mathbf{BA}^2]=[\mathbf{T},(\mathbf{BA})^2]=[\mathbf{T},\mathbf{A}^{-1}\mathbf{BA}]=[\mathbf{A}^2,\mathbf{TB}]=0$$
$$(\mathbf{ATB})+(\mathbf{ATB})^{-1}=2,\quad \mathbf{A}^6=-1\ .$$

It is easy to see that the PF solutions of the previous section are compatible with the above monodromies.

6. Discussion of Model B

We would like to discuss how Model B specializes to the rigid $SU(3)$, $SU(2)\times SU(2)$ and $SU(2)$ $N=2$ Yang-Mills theories, respectively, near the appropriate points in moduli space; we will mainly be concerned with the $SU(3)$ point, but will also briefly discuss the two other cases.

On the heterotic side, Model B is obtained by compactifying the $E_8\times E_8$ heterotic string on $K3\times T^2$ with the second Chern class of the gauge bundle chosen to be $(12,12)$. This model has three vector multiplets containing S,T,U, and 244 hypermultiplets. Note that in perturbation theory we get in this model an $SU(2)$ enhanced gauge symmetry on the line $T=U$, an $SU(2)\times SU(2)$ gauge symmetry at the point $T=U=i$, and an $SU(3)$ gauge symmetry at the point $T=U=\rho\equiv e^{2\pi i/3}$ [29,20,21,12].

In the dual type II string framework [7], the defining polynomial of the Calabi-Yau manifold is:

$$p=\frac{1}{24}{z_1}^{24}+\frac{1}{24}{z_2}^{24}+\frac{1}{12}{z_3}^{12}+\frac{1}{3}{z_4}^{3}+\frac{1}{2}{z_5}^{2}-\psi_0 z_1z_2z_3z_4z_5-\frac{1}{6}\psi_1(z_1z_2z_3)^6-\frac{1}{12}\psi_2(z_1z_2)^{12} \qquad (6.1)$$

in $WP^{24}_{1,1,2,8,12}$. Suitable variables are: $x=-\psi_0^6/\psi_1$, $y=1/\psi_2^2$ and $z=-\psi_2/\psi_1^2$, in terms of which the discriminant is

$$\Delta\ =\ \left(y-1\right)\Big\{\left(1-z\right)^2-y\,z^2\Big\}\Big\{\left(\left(1-x\right)^2-z\right)^2-\,y\,z^2\Big\}\ \equiv:\ \Delta_y\,\Delta_z\,\Delta_x\ . \qquad (6.2)$$

We are mainly interested in the region of moduli space near the zero of Δ at $x=0$, $y=0$ and $z=1$, which describes the point of enhanced $SU(3)$ symmetry. We first need to find the relationship between x,z and the variables u,v of rigid $SU(3)$ Yang-Mills theory, in the

semi-classical domain where $y=0$. For this, we can make use of the following identification of x, z with modular functions [13,15]:

$$
\begin{aligned}
(x)^{-1} &= 864\frac{j(T)+j(U)-1728}{j(T)j(U)+\sqrt{j(T)(j(T)-1728)}\sqrt{j(U)(j(U)-1728)}} \\
z &= \frac{(j(T)j(U)+\sqrt{j(T)(j(T)-1728)}\sqrt{j(U)(j(U)-1728)})^2}{j(T)j(U)(j(T)+j(U)-1728)^2} \ ,
\end{aligned}
\tag{6.3}
$$

where T, U are the flat coordinates associated with x, z. Specifically, we can expand the j-functions near the origin as follows:

$$
\begin{aligned}
j(T) &= c\left(\frac{T-\rho}{T-\rho^2}\right)^3 \equiv (\sqrt{\alpha'}a_T)^3 \\
j(U) &= c\left(\frac{U-\rho}{U-\rho^2}\right)^3 \equiv (\sqrt{\alpha'}a_U)^3 \ ,
\end{aligned}
\tag{6.4}
$$

where c is some constant that we will neglect in the following. The point is that a_T, a_U have a simple relationship to the $SU(3)$ Cartan subalgebra variables, a_1, a_2 [12]:

$$
a_T = \rho\, a_1 + a_2 \ , \qquad a_U = -(\rho^2\, a_1 + a_2) \ . \tag{6.5}
$$

This can be seen by comparing the Weyl- and modular transformation behavior, and noting that $j=0$ $(T, U=\rho)$ corresponds to the fixed point of the modular transformation $\mathbf{ST}$. In particular, the combined Z_3 transformations $\mathbf{ST}$ (acting on T) and $(\mathbf{ST})^{-1}$ (acting on U) induce a Coxeter transformation on a_i, and $T \leftrightarrow U$ yields a Weyl reflection. The three lines in the CSA on which there is an unbroken $SU(2)\times U(1)$ are given by $a_T = (1,\rho,\rho^2)\cdot a_U$ [12].

By taking a_T, a_U large (but α' sufficiently small so that $\sqrt{\alpha'}a_{T,U} \ll 1$), we can now make use of the semi-classical relationship between a_i and the Casimirs: $u = {a_1}^2 + {a_1}^2 - a_1 a_2$, $v = a_1 a_2(a_1 - a_2)$, and thus express x, z in terms of u, v via the j-functions. In this way, we are led to write

$$
\begin{aligned}
x &= 2\,(\alpha' u)^{3/2} \\
y &= 27(\alpha')^3{\Lambda_S}^6 \equiv: \epsilon^2 \\
z &= 1-(\alpha')^{3/2}\left(2\,u^{3/2}+3\sqrt{3}\,v\right) \ ,
\end{aligned}
\tag{6.6}
$$

where $\Lambda_S \equiv \Lambda\exp(-\hat{S}/6)$. Indeed, in terms of these variables, the leading piece of the discriminant (6.2) in the $\alpha' \to 0$ limit is given precisely by the discriminant of SU(3) quantum Yang-Mills theory [4,5],

$$
\Delta_{SU(3)} = (4\,u^3 - 27\,(v-{\Lambda_S}^3)^2)\,(4\,u^3-27\,(v+{\Lambda_S}^3)^2) \ . \tag{6.7}
$$

We now introduce the dimensionless quantities $\tilde{u} = u/(27^{1/6}\Lambda_S)^2$, $\tilde{v} = v/(27^{1/6}\Lambda_S)^3$, and consider the following variables which correspond to blowing up the singular point $x = y = 0$, $z = 1$ in a particular way:

$$
\begin{aligned}
x_1 &= \frac{y}{x^2} = \frac{1}{4\tilde{u}^3} \\
x_2 &= \frac{1}{x}\big(1 - x - z\big) = \sqrt{\frac{27\tilde{v}^2}{4\tilde{u}^3}} \\
x_3 &= \frac{1}{2}x = \epsilon\,\tilde{u}^{3/2}\,.
\end{aligned} \tag{6.8}
$$

In terms of these variables, and after rescaling the solutions by ${x_1}^{1/6}\sqrt{x_3}$, the Picard-Fuchs operators (given in [18]) take the following form:

$$
\begin{aligned}
\mathcal{L}_1 &= x_3 + 24x_1\,(5x_3 - 4)\,\partial_1 + 72{x_1}^2\,(2x_3 - 1)\,{\partial_1}^2 - 6\,(5x_2 + 6x_3 + 6x_2x_3)\,\partial_2 \\
&\quad - 72x_1x_2\partial_1\partial_2 + 18\,(1 + x_2)\,(1 - x_2 - 2x_3 - 2x_2x_3)\,{\partial_2}^2 + 6x_3\,(8x_3 - 1)\,\partial_3 \\
&\quad + 72x_1\,(1 - 2x_3)\,x_3\partial_1\partial_3 + 36x_2x_3\partial_2\partial_3 + 18{x_3}^2\,(2x_3 - 1)\,{\partial_3}^2 \\
\mathcal{L}_2 &= 16x_1{x_3}^2 - 1 + 24x_1\left(11x_1{x_3}^2 - 2\right)\partial_1 + 36{x_1}^2\left(4x_1{x_3}^2 - 1\right){\partial_1}^2 \\
&\quad + 12x_1x_3\,(1 - 2x_3 - 2x_2x_3)\,\partial_2 + 72{x_1}^2x_3\,(1 - 2x_3 - 2x_2x_3)\,\partial_1\partial_2 + \\
&\quad 9x_1(2x_3 + 2x_2x_3 - 1)^2{\partial_2}^2 \\
\mathcal{L}_3 &= 7 + 48x_1\partial_1 + 144{x_1}^2{\partial_1}^2 + 84\,(1 + x_2)\,\partial_2 - 144x_1\,(1 + x_2)\,\partial_1\partial_2 \\
&\quad + 36(1 + x_2)^2{\partial_2}^2 + 84x_3\partial_3 - 144x_1x_3\partial_1\partial_3 + 72\,(x_3 + x_2x_3 - 1)\,\partial_2\partial_3 \\
&\quad + 36{x_3}^2{\partial_3}^2\,.
\end{aligned} \tag{6.9}
$$

For $x_3 \to 0$, these operators turn precisely into the PF operators of the rigid SU(3) theory ! Recall [26] that these are given by an Appell system of type $F_4(a, b; c, c'; \alpha, \beta)$ of the form

$$
\begin{aligned}
\tilde{\mathcal{L}}_1 &= \theta_\alpha(\theta_\alpha + c - 1) - \alpha(\theta_\alpha + \theta_\beta + a)(\theta_\alpha + \theta_\beta + b) \\
\tilde{\mathcal{L}}_2 &= \theta_\beta(\theta_\beta + c' - 1) - \beta(\theta_\alpha + \theta_\beta + a)(\theta_\alpha + \theta_\beta + b)
\end{aligned} \tag{6.10}
$$

(where $\theta_\alpha \equiv \alpha\partial_\alpha$ etc.), with $(a, b; c, c'; \alpha, \beta) = (-1/6, -1/6; 2/3, 1/2; 1/x_1, {x_2}^2/x_1)$. It immediately follows that the period vector has for $\alpha' \to 0$ the rigid $SU(3)$ periods $a_i, a_{D,i}$, $i = 1, 2$ among its components, and thus that the exact string theory prepotential indeed contains the non-perturbative prepotential [26] of the rigid $SU(3)$ theory. Note that the form of the variable x_3 reflects that, analogous to model A, string corrections break the global Z_{12} R-symmetry of the rigid Yang-Mills theory down to Z_6.

Though we do not want to go into the details, we just note that we find for the PF operators $\mathcal{L}_i$ four series and four logarithmic solutions, each with indices $(-1/6, 0, -1/3)$, $(-1/6, 0, 1/3)$, $(-1/6, 0, 0)$ and $(-1/6, 1, 0)$. The series and logarithmic solutions with the last two indices contain the rigid $SU(3)$ periods. Specifically, after undoing the rescaling by ${z_1}^{1/6}\sqrt{z}_3$, we find a behavior of the PF solutions that is fully compatible with a period vector of the form $(1, S, \sqrt{\alpha'}a_1, \sqrt{\alpha'}a_2, \sqrt{\alpha'}a_{D,1}, \sqrt{\alpha'}a_{D,2}, \alpha' u, \alpha' uS)$; similar to the previous discussion, u is a period due to the fact[10] that $u = -2\pi i \partial_S F_{SW}^{SU(3)}$.

There are scaling arguments for model B that are similar to the one for model A, with however a somewhat surprising conclusion. That is, near the $SU(3)$ singular point the moduli scale as:

$$\begin{aligned} \psi_0 &= -2^{1/6}\epsilon^{1/12}\tilde{u}^{1/4} \\ \psi_1 &= -\frac{1}{\sqrt{\epsilon}} - \sqrt{\epsilon}\Big(\tilde{u}^{3/2} + \frac{3}{2}\sqrt{3}\tilde{v}\Big) \\ \psi_2 &= -\frac{1}{\epsilon} \end{aligned} \tag{6.11}$$

Blowing up the singular neighborhood requires rescaling $z_3 \to \epsilon^{-1/12} z_3$, and after further irrelevant numerical rescalings and setting $\zeta = z_1 z_2$, we arrive at the following representation of (6.1) near the singularity:

$$\begin{aligned} p_0 &= \frac{1}{12}\big(\zeta^{12} + 2\zeta^6 {z_3}^6 + {z_3}^{12}\big) \;=\; \epsilon \\ p_1 &= \frac{1}{24}{z_1}^{24} + \frac{1}{24}{z_2}^{24} + \frac{1}{3}{z_4}^3 + \frac{1}{2}{z_5}^2 \\ &\quad - \big(\frac{1}{6}\tilde{u}^{3/2} + \frac{\sqrt{3}}{4}\tilde{v}\big){z_1}^{12}{z_2}^{12} - 2^{1/6} i \tilde{u}^{1/4} {z_1}^2 {z_2}^2 z_4 z_5 = -1 \;, \end{aligned} \tag{6.12}$$

where we have already eliminated z_3 from p_1 via solving $p_0 = 0$. Like for model A, $p_0 = 0$ is a singular, rigid manifold, and the sub-leading piece, $p_1 = -1$, carries all moduli dependence. However, the similarity stops there, in that $p_0 = 0$ does not correspond to a singular $K3$, but to a singular point. Furthermore, $p_1 = -1$ can be compactified to a Calabi-Yau manifold by adding an extra weight 2 variable w, i.e. by considering $p_1 + w^{12} = 0$ in the same weighted projective space as the original Calabi-Yau. This "rigid" Calabi-Yau manifold has no obvious relationship to the hyperelliptic, genus two curve [4,5] in $WP^6_{1,1,3}$ that underlies the $SU(3)$ gauge theory!

[10] This can be shown for all $SU(n)$ [30].

Indeed, expecting to see the genus two curve is perhaps too much to ask for[11]. All that is required is that the Calabi-Yau $p_1+w^{12}=0$ reproduces the $SU(3)$ quantum discriminant (which it does), and, more specifically, that it has $(a_i, a_{D,i})$ among its periods. In other words, the sub-leading piece in the degeneration of (6.1) is required to be equivalent to the hyperelliptic curve only as far as the variation of Hodge structure is concerned, since this is the only attribute that is directly relevant for physical computations. Following reasoning similar to that which led us to the meromorphic one-form for model A, we expect that periods of a meromorphic three-form (obtained by multiplying the holomorphic three-form of $p_1+w^{12}=0$ by $w^5/(z_1^5 z_2^5)$) provide an alternative description of $SU(3)$ quantum $N=2$ Yang-Mills theory.

We can get additional insight in the geometry of the specialization, by studying the location of the nodes on the Calabi–Yau manifold in relation to the other two types of semi-classical gauge symmetry enhancement. These are characterized by the $SU(2)$ line: $T=U \longleftrightarrow (\Delta_z=0,\ \Delta_x\neq 0)$, and by the $SU(2)\times SU(2)$ point: $T=U=i \longleftrightarrow (\Delta_z=0,\ \Delta_x=0,\ x=2)$. A rough sketch of the relevant part of the moduli space is given in Fig.2. Specifically, the coordinates of the nodes on the Calabi–Yau are

$$
\begin{aligned}
&a)\ \Delta_z=0:\ (1,1,\psi_1^{1/6},0,0)\\
&b)\ \Delta_x=0:\ (1,1,(\psi_0^6+\psi_1)^{1/6},\psi_0^2(\psi_0^6+\psi_1)^{1/3},\psi_0^3(\psi_0^6+\psi_1)^{1/2})\ ,
\end{aligned}
\tag{6.13}
$$

up to equivalences induced by phase symmetries (we have displayed only one of two branches).

For the $SU(2)\times SU(2)$ point, the relation between heterotic and Calabi–Yau moduli can, in leading order, be inferred from: $j(T)=1728+432\epsilon a_T^2,\ j(U)=1728+432\epsilon a_U^2$, with $y=\epsilon^2$ and $a_T=a_1+a_2$ and $a_U=a_1-a_2$. If we proceed as before and rescale the coordinates in order to make the location of the nodes independent of the limit $\epsilon\to 0$, we find:

$$
\begin{aligned}
p_0 &= \frac{1}{12}y^{12}+\frac{1}{12}{z_3}^{12}+\frac{1}{3}{z_4}^{3}+\frac{1}{2}{z_5}^{2}+\frac{1}{6}y^6{z_3}^6-(2\alpha)^{1/6}y\,z_3z_4z_5\\
p_1 &= \frac{1}{24}{z_1}^{24}+\frac{1}{24}{z_2}^{24}+\frac{u_2}{12}(z_1z_2z_3)^6-\frac{1}{24}2^{1/6}\alpha^{-5/6}(u_1+u_2)z_1z_2z_3z_4z_5\ ,
\end{aligned}
\tag{6.14}
$$

[11] Even for model A one can obtain, instead of the Seiberg-Witten torus, a “rigid” Calabi-Yau in the original weighted projective space that gives the same periods with an appropriate meromorphic three-form.

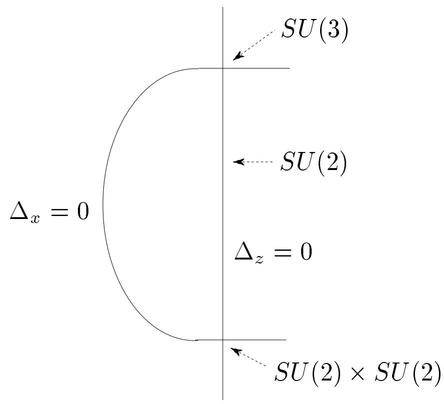

Fig.2 *Shown is the singular locus for $y = 0$, on which certain three-cycles shrink to zero size and where various monopoles become massless. We indicated the regions that are associated with semi-classical gauge symmetry enhancements. Note that the line $\Delta_x = 0$ is not related to a semi-classical gauge symmetry, but rather is a pure quantum effect.*

where we have set $y = z_1 z_2$ and where we have used the semi-classical relationship $u_i = a_i^2$. Specifically, the parameter α takes a value equal to one, and it is precisely for this value that the $K3$ surface develops *two* singularities, corresponding to the nodes a) and b) (6.13) of the Calabi–Yau manifold. Inserting the coordinates of the singularities into p_1 as before, we get two branches associated with

$$p_1(i) = \frac{1}{24} z_1^{24} + \frac{1}{24} z_2^{24} + \frac{1}{12} u_i \; z_1^{12} z_2^{12} \;, \qquad i = 1, 2 \;.$$

That is, we obtain two copies of a six-fold cover of the Seiberg–Witten torus (3.5). Note that the nodes are located at points that are separated by an infinite distance when we take the global limit. It is plausible that it is this infinite distance between the nodes in the global limit that leads to the two decoupled $SU(2)$ factors.

For a generic point on the $SU(2)$ line, where $\Delta_z = 0$, node b) is not developed and indeed, following similar reasoning, we find just a single copy of the $SU(2)$ curve. As we approach the $SU(3)$ point, node b) develops, but in this case without the rescaling of z_4, z_5 that infinitely separated the nodes at the $SU(2) \times SU(2)$ point. The $SU(3)$ periods will arise in part from integration contours that link both nodes, and apparently it is due to the finite separation of the two nodes that we do not find a genus two curve in a simple way. More generally, adopting this point of view suggests that while the rank of the gauge

group is given by the total number of nodes, the rank of a simple group factor equals the number of a given set of nodes that are not infinitely separated as $\alpha' \to 0$.

Finally we note that it is easy to check that appropriate meromorphic forms for the tori come out in a way that is quite similar to what we have discussed in section 3.

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